Notes on Doi-Peliti Field Theory

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1 Introduction

Doi-Peliti field theory is a framework for analyzing stochastic processes by mapping them to field theories expressed as path integrals. The approach begins by transforming a master equation into a quantum-mechanics problem using creation and annihilation operators, which can then be reformulated as a path integral. These transformations are detailed in the following notes.

My interest was piqued by the paper *Field-theoretic approach to fluctuation effects in neural networks* by Buice and Cowan (PRE, 2007), which uses this framework to study a neural-network model where neurons emit varying numbers of spikes over time. The network dynamics, governed by the evolution of the joint probability distribution of spike numbers across neurons, follows a master equation. The mean-field behavior of these dynamics yields a version of the Wilson-Cowan equations, while the Doi-Peliti path integral enables the calculation of fluctuations around this solution.

Link to Buice and Cowan: `https://journals.aps.org/pre/abstract/10.1103/PhysRevE.75.051919`

These notes are based on several excellent resources:

- Notes by Johannes Pausch: `https://johannespausch.github.io/NESFT-Pausch-LectureNotes2020.pdf`
- Notes by Gunnar Pruessner: `https://www.ma.imperial.ac.uk/~pruess/publications/Gunnar_Pruessner_field_theory_notes.pdf`
- Notes by John Cardy (see *Field Theory and Non-Equilibrium Statistical Mechanics* PostScript notes): `https://www-thphys.physics.ox.ac.uk/people/JohnCardy/`
- Chapter 9 ("Reaction-diffusion systems") of the textbook *Critical Dynamics* by Uwe C. Täuber: `https://www.cambridge.org/core/books/critical-dynamics/041557627C8F8F36D96084B7617BFD5D`

The Appendix of Buice and Cowan also provides an instructive introduction to this formalism.

2 Master Equation

Consider a natural number $N(t) \in \{0, 1, \ldots\}$ that varies in continuous time, described by a probability distribution $P(N(t_1), \ldots, N(t_n))$ for any times $t_1, \ldots, t_n$. We assume only that this process is **Markovian**, which means that, if $t_1 < \ldots < t_n$,

$$P(N(t_n)|N(t_1), \ldots, N(t_{n-1})) = P(N(t_n)|N(t_{n-1})). \tag{1}$$

In other words, conditioning on past observations can be restricted to the most recent observation.

The Markov property allows us to derive a **master equation** that governs the evolution of the conditional probability $P(N(t) = n|N(0) = n_0)$ for $n \in \{0, 1, \ldots\}$. We begin by examining $P(N(t + \Delta t) = n|N(0) = n_0)$, which we can write as

$$\begin{aligned}
&P(N(t + \Delta t) = n|N(0) = n_0) \\
&= \sum_m P(N(t + \Delta t) = n|N(t) = m, N(0) = 0)P(N(t) = m|N(0) = n_0) \quad \text{(probability chain rule)} \\
&= \sum_m P(N(t + \Delta t) = n|N(t) = m)P(N(t) = m|N(0) = n_0) \quad \text{(Markov property)} \\
&= \sum_m \left[\underbrace{P(N(t) = n|N(t) = m)}_{=\delta_{nm}} + \Delta t \underbrace{\partial_{t'} P(N(t') = n|N(t) = m)\big|_{t'=t}}_{=W_t(n|m)} \right] P(N(t) = m|N(0) = n_0).
\end{aligned} \tag{2}$$

The only approximation is the last step, where we expand to first order in Δt. Here we introduce the **transition rate** $W_t(n|m)$, which is the instantaneous rate of change of the probability of transitioning from state m to state n at time t. Due to probability conservation, these rates must satisfy

$$\sum_{n=0}^{\infty} W_t(n|m) = 0 \tag{3}$$

for all m, which leads to

$$W_t(n|n) = -\sum_{m \neq n} W_t(m|n). \tag{4}$$

In Eq. (2), separating the $m = n$ and $m \neq n$ cases in the sum and using Eq. (4) gives

$$\begin{aligned}
&\frac{1}{\Delta t} \Big[P(N(t + \Delta t) = n|N(0) = n_0) - P(N(t) = n|N(0) = n_0) \Big] \\
&\qquad = \sum_{m \neq n} \Big[W_t(n|m)P(N(t) = m|N(0) = n_0) - W_t(m|n)P(N(t) = n|N(0) = n_0) \Big].
\end{aligned} \tag{5}$$

Taking $\Delta t \to 0$ yields the master equation

$$\boxed{\partial_t P(N(t) = n|N(0) = n_0) = \sum_{m \neq n} \left[W_t(n|m) P(N(t) = m|N(0) = n_0) - W_t(m|n) P(N(t) = n|N(0) = n_0) \right].} \tag{6}$$

The right-hand side has two terms:

- A **gain** term representing the increase in probability due to transitions into state n from other states m
- A **loss** term representing the decrease in probability due to transitions out of state n into other states m

Note that this derivation did not use the fact that $N(t)$ is a natural number—the state space is completely arbitrary as far as Eq. (6) is concerned. Going forward, however, we will focus on the case of a single natural number. Extension to the case of several natural numbers on a lattice (modeling, for example, a network of spiking neurons arranged in space) is straightforward.

Extinction Process

As an illustrative example, consider an extinction process where $N(t)$ is the number of particles present at time t, with each particle having a decay rate ϵ. The master equation for this process is

$$\frac{\partial}{\partial t} P(N(t) = n|N(0) = n_0) = \epsilon \left[(n+1) P(N(t) = n+1|N(0) = n_0) - n P(N(t) = n|N(0) = n_0) \right]. \tag{7}$$

On the right-hand side:

- The term $dt\epsilon(n+1)$ is the probability that, within time interval dt, one of $n+1$ particles present decays, resulting in a state with n particles (gain term)
- The term $dt\epsilon n$ is the probability that, within time interval dt, one of the n particles present decays, resulting in a state with $n-1$ particles (loss term)

3 Quantum Mechanical Formulation

Note that the master equation describes linear dynamics (potentially with a time-dependent dynamics matrix, if the transition rates are time-dependent) of a vector with components $P(N(t) = n)$ for $n \in \{0, 1, \ldots\}$. This is analogous to quantum mechanics. We now introduce quantum mechanical notation inspired by this analogy.

Creation/annihilation operators and number states

We begin by defining three objects: the **creation operator** $a^\dagger$, its Hermitian conjugate the **annihilation operator** a, and a special state called the **vacuum** $|0\rangle$. These satisfy

$$[a, a^\dagger] = 1, \tag{8a}$$

$$a|0\rangle = 0, \tag{8b}$$

$$\langle 0|0\rangle = 1. \tag{8c}$$

We then define the **number states** $|n\rangle$ as

$$|n\rangle = (a^\dagger)^n |0\rangle . \tag{9}$$

From the properties in Eq. (8), we can derive[1]

$$a^\dagger |n\rangle = |n+1\rangle , \tag{10}$$

$$a|n\rangle = n|n-1\rangle , \tag{11}$$

$$\langle n|m\rangle = n!\delta_{nm}. \tag{12}$$

The notes of Pruessner keep Eqs. (10) and (11) but enforce $\langle n|m\rangle = \delta_{nm}$, which results in a and $a^\dagger$ not being Hermitian conjugates of each other.

The **number operator** $a^\dagger a$ counts the number of particles through the relation

$$a^\dagger a|n\rangle = n|n\rangle . \tag{13}$$

Wavefunction and pseudo-Hamiltonian

Unlike quantum mechanics where the wavefunction represents the square root of probability, we define our wavefunction to be *linear* in probability (because the master equation is linear in

[1] Note that our definition of number states differs from standard quantum mechanical conventions. In terms of our definition, quantum mechanics uses the normalization

$$|n\rangle_{\text{QM}} = \frac{1}{\sqrt{n!}}|n\rangle ,$$

which leads to

$$a^\dagger |n\rangle_{\text{QM}} = \sqrt{n+1}\,|n+1\rangle_{\text{QM}}$$

$$a|n\rangle_{\text{QM}} = \sqrt{n}\,|n-1\rangle_{\text{QM}}$$

$${}_{\text{QM}}\langle n|m\rangle_{\text{QM}} = \delta_{nm}$$

with the same commutation relations and vacuum-state normalization as in Eq. (8).

probability):

$$|\psi(t)\rangle = \sum_n P(N(t) = n)\,|n\rangle\,. \tag{14}$$

In this representation, the master equation takes the form

$$\partial_t\,|\psi(t)\rangle = -H(a^\dagger, a)\,|\psi(t)\rangle\,, \tag{15}$$

where $H(a^\dagger, a)$ is the **pseudo-Hamiltonian** ("pseudo" because there is no i on the left-hand side; this is the terminology used by Täuber), also called the Liouvillian. We assume $H(a^\dagger, a)$ can be expressed as a polynomial of creation and annihilation operators. We also assume that it is in **normal-ordered form**, meaning all creation operators appear to the left of annihilation operators. Any polynomial in $a^\dagger$ and a can be normal-ordered using the commutation relations from Eq. (8). Importantly, we have assumed that the transition rates in the original master equation are time-independent so that $H(a^\dagger, a)$ is time-independent. The solution to Eq. (15) is

$$|\psi(t)\rangle = e^{-H(a^\dagger,a)t}\,|\psi(0)\rangle\,. \tag{16}$$

Probabilities and expectation values

We can extract probabilities from the wavefunction using

$$P(N(t) = n) = \frac{1}{n!}\,\langle n|\psi(t)\rangle\,. \tag{17}$$

For expectation values of the form

$$\langle \mathcal{O}(t)\rangle = \sum_{n=0}^{\infty} P(N(t) = n)c_n, \tag{18}$$

where c_n are numbers defining the observable, we encode these numbers as eigenvalues of an operator

$$\mathcal{O} = \sum_{n=0}^{\infty} c_n \frac{|n\rangle\langle n|}{n!}, \tag{19}$$

which gives us

$$\langle n|\psi(t)\rangle\, c_n = \langle n|\,\mathcal{O}\,|\psi(t)\rangle\,. \tag{20}$$

This leads to the formula

$$\begin{aligned}
\langle \mathcal{O}(t) \rangle &= \sum_{n=0}^{\infty} P(N(t)=n) c_n \quad \text{(restating Eq. 18)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \langle n|\psi(t)\rangle \, c_n \quad \text{(using Eq. (17))} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \langle n| \mathcal{O} |\psi(t)\rangle \quad \text{(using Eq. (20))} \\
&= \langle 0| \sum_{n=0}^{\infty} \frac{a^n}{n!} \mathcal{O} |\psi(t)\rangle . \qquad (21)
\end{aligned}$$

We introduce the bra vector

$$\langle 0| e^a, \quad \text{where} \quad e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}, \qquad (22)$$

which projects a state onto each number state $\langle n|$, normalizes by dividing by $n!$, and sums over n. We then obtain the elegant result

$$\boxed{\langle \mathcal{O}(t) \rangle = \langle 0| e^a \mathcal{O} |\psi(t)\rangle .} \qquad (23)$$

Probabilities summing to unity is equivalent to

$$\langle 0| e^a |\psi(t)\rangle = 1. \qquad (24)$$

4 Coherent states

Exponentiating creation and annihilation operators yields important eigenvector-eigenvalue relations. For any complex number ϕ, we have

$$\langle 0| e^{\phi^* a} a^\dagger = \langle 0| e^{\phi^* a} \phi^*, \qquad (25)$$

$$a e^{\phi a^\dagger} |0\rangle = \phi e^{\phi a^\dagger} |0\rangle , \qquad (26)$$

where Eq. (26) is the Hermitian conjugate of Eq. (25). These relations can be obtained by expanding the exponential in a Taylor series as in Eq. (22). The kets $e^{\phi a^\dagger} |0\rangle$ form a set of right eigenvectors of a with eigenvalue ϕ, and likewise the bras $\langle 0| e^{\phi^* a}$ form a set of left eigenvectors of $a^\dagger$ with eigenvalue ϕ^*. A special case of the latter is the bra $\langle 0| e^a$ from Eq. (22), which is a left eigenvector of $a^\dagger$ with eigenvalue unity. The fact that these eigenvectors are parameterized by a complex number rather than an integer indicates that they form an overcomplete set. These eigenvectors are called **coherent states**.

These coherent states provide a **resolution of the identity**,

$$1 = \int \frac{d\phi^* \wedge d\phi}{2\pi i} e^{-|\phi|^2} e^{\phi a^\dagger} |0\rangle\langle 0| e^{\phi^* a}. \tag{27}$$

The wedge product in the measure is a convenient way to denote integration over the complex plane. For $\phi = x + iy$ (where x and y are real), we have

$$\frac{d\phi^* \wedge d\phi}{2i} = \frac{(dx - idy) \wedge (dx + idy)}{2i} = dx \wedge dy, \tag{28}$$

using the antisymmetry property of the wedge product, $A \wedge B = -B \wedge A$. The resulting integral $\int dx \wedge dy(\cdots)$ is evaluated as $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy(\cdots)$. When Eq. (27) is written in polar coordinates in the complex plane, one can show that it becomes $\sum_{n=0}^{\infty} \frac{1}{n!} |n\rangle\langle n|$, which is the identity operator because the number states form an orthonormal basis (when normalized by $1/\sqrt{n!}$). The factor $e^{-|\phi|^2}$ in Eq. (27) can be thought of as compensating for the overcompleteness of the coherent states.

5 Probability conservation

A property of the pseudo-Hamiltonian is that setting $a^\dagger = 1$ results in $H(1, a) = 0$. This results from probability conservation over time, which is equivalent to, for all t,

$$\langle 0| e^a e^{-H(a^\dagger, a)t} |\psi(0)\rangle = 1 \tag{29}$$

Taylor-expanding the left-hand side in t yields

$$\sum_{n=1}^{\infty} \langle 0| e^a \left(H(a^\dagger, a)\right)^n |\psi(0)\rangle \frac{(-t)^n}{n!} = 0, \tag{30}$$

where we note that the $n = 0$ term gives $\langle 0| e^a |\psi(0)\rangle = 1$ for a normalized initial wavefunction (Eq. (24)). Each term $n \geq 1$ in the sum must individually vanish. Taking the $n = 1$ term and using normal ordering, along with the fact that $\langle 0| e^a$ is a left eigenvector of $a^\dagger$ with eigenvalue unity (Sec. 4), we have

$$\langle 0| e^a H(a^\dagger, a) |\psi(0)\rangle = \langle 0| e^a H(1, a) |\psi(0)\rangle = 0. \tag{31}$$

Since $|\psi(0)\rangle$ is arbitrary, we have $\langle 0| e^a H(1, a) = 0$. Since e^a can be commuted past $H(1, a)$ and is invertible, we have $\langle 0| H(1, a) = 0$. Finally, since $H(1, a)$ contains only annihilation operators which create particles when acting on the vacuum bra to its left, we arrive at

$$\boxed{H(1, a) = 0.} \tag{32}$$

In quantum mechanics, one can in principle use any (Hermitian) operator as a Hamiltonian. By contrast, pseudo-Hamiltonians used in Doi-Peliti theory satisfy the special constraint Eq. (32) due to their origin in probability-conserving master equations.

A consequence of Eq. (32) is that, for all t,

$$\langle 0| e^{a} e^{-H(a^\dagger,a)t} = \langle 0| e^{a}. \tag{33}$$

6 Path integral

Consider evolving the vacuum state $|0\rangle$ under the pseudo-Hamiltonian $H(a^\dagger, a)$ from time t_i to t_f. The expectation value of an observable $\mathcal{O}$ at an intermediate time t can be written as

$$\langle \mathcal{O}(t)\rangle = \langle 0| e^{a} e^{-H(a^\dagger,a)(t_f-t)} \mathcal{O}(a^\dagger, a) e^{-H(a^\dagger,a)(t-t_i)} |0\rangle . \tag{34}$$

We *could* simplify Eq. (34) using Eq. (33), i.e.,

$$\langle 0| e^{a} e^{-H(a^\dagger,a)(t_f-t)} = \langle 0| e^{a}, \tag{35}$$

however, retaining the time evolution from t to t_f allows for generalization to inserting additional operators to compute correlation functions in the path integral.

We do, however, simplify Eq. (34) using the commutation relation

$$e^{a} a^\dagger = (a^\dagger + 1) e^{a} \quad \text{(i.e., } [e^{a}, a^\dagger] = e^{a}). \tag{36}$$

Using this relation, we can commute e^{a} past all operators in Eq. (34), replacing each $a^\dagger$ with $a^\dagger + 1$. We can then eliminate e^{a} altogether by noting that $e^{a} |0\rangle = |0\rangle$ (although this elimination is a consequence of our simplifying assumption that the initial state is the vacuum; in general, e^{a} must be absorbed into the initial state). This operation of replacing $a^\dagger$ with $a^\dagger + 1$ is known as the **Doi shift**. It yields

$$\langle \mathcal{O}(t)\rangle = \langle 0| e^{-H(a^\dagger+1,a)(t_f-t)} \mathcal{O}(a^\dagger + 1, a) e^{-H(a^\dagger+1,a)(t-t_i)} |0\rangle . \tag{37}$$

To rewrite this as a path integral, we introduce a time-indexed version of the resolution of the identity (Eq. (27)),

$$1_t = \int \frac{d\phi^*(t) \wedge d\phi(t)}{2\pi i} e^{-|\phi(t)|^2} e^{\phi(t) a^\dagger} |0\rangle\langle 0| e^{\phi^*(t) a}. \tag{38}$$

Eq. (37) can be expressed as

$$\langle \mathcal{O}(t)\rangle = \langle 0|\ 1_{t_f}\ E_{\mathrm{DS}}\ 1_{t_f-\Delta t}\ \cdots\ 1_{t+\Delta t}\ E_{\mathrm{DS}}\ 1_t\ \mathcal{O}_{\mathrm{DS}}\ 1_{t-\Delta t}\ E_{\mathrm{DS}}\ 1_{t-2\Delta t}\ \cdots\ 1_{t_i}\ E_{\mathrm{DS}}\ 1_{t_i-\Delta t}\ |0\rangle , \tag{39}$$

$$E_{\mathrm{DS}} = e^{-\Delta t H(a^\dagger+1,a)}, \tag{40}$$

$$\mathcal{O}_{\mathrm{DS}} = \mathcal{O}(a^\dagger + 1, a). \tag{41}$$

Defining the measure

$$\mathcal{D}\phi = \frac{d\phi^*(t_f) \wedge d\phi(t_f)}{2\pi i} \cdots \frac{d\phi^*(t_i - \Delta t) \wedge d\phi(t_i - \Delta t)}{2\pi i}, \tag{42}$$

we can write

$$\begin{aligned} \langle \mathcal{O}(t) \rangle = \int \mathcal{D}\phi \exp \Bigg(& - |\phi(t_f)|^2 - \cdots - |\phi(t_i)|^2 - |\phi(t_i - \Delta t)|^2 \\ & + \log \langle 0| e^{\phi^*(t_f)a} E_{\mathrm{DS}} e^{\phi(t_f - \Delta t)a^\dagger} |0\rangle + \cdots \\ & + \log \langle 0| e^{\phi^*(t_i)a} E_{\mathrm{DS}} e^{\phi(t_i - \Delta t)a^\dagger} |0\rangle \Bigg) \mathcal{O}_{\mathrm{DS}}(\phi^*(t), \phi(t)), \end{aligned} \tag{43}$$

where

$$\mathcal{O}_{\mathrm{DS}}(\phi^*(t), \phi(t)) = \mathcal{O}(\phi^*(t) + 1, \phi(t)). \tag{44}$$

Each term of the form $\log \langle 0| e^{\phi^*(t)a} E_{\mathrm{DS}} e^{\phi(t-\Delta t)a^\dagger} |0\rangle$ is some function of $\phi^*(t)$ and $\phi(t - \Delta t)$. Note that the variable $\phi^*(t_i - \Delta t)$ appears only once in the exponential of Eq. (43), namely, in the term $|\phi(t_i - \Delta t)|^2$. We can therefore integrate it out, yielding a delta function that enforces $\phi_(t_i - \Delta t) = 0$, reflecting the initial state being the vacuum.

Working to linear order in Δt, we have

$$\log \langle 0| e^{\phi^*(t)a} E_{\mathrm{DS}} e^{\phi(t-\Delta t)a^\dagger} |0\rangle = \phi^*(t)\phi(t - \Delta t) - \Delta t H_{\mathrm{DS}}(\phi^*(t), \phi(t - \Delta t)) + \mathcal{O}(\Delta t^2), \tag{45}$$

where

$$H_{\mathrm{DS}}(\phi^*(t), \phi(t - \Delta t)) = H(\phi^*(t) + 1, \phi(t - \Delta t)). \tag{46}$$

Taking the limit $\Delta t \to 0$ yields the Doi-shifted path integral:

$$\langle \mathcal{O}(t) \rangle = \int \mathcal{D}\phi \exp \left(- \int_{t_i}^{t_f} dt' \Big(\phi^*(t')\partial_{t'}\phi(t') + H_{\mathrm{DS}}(\phi^*(t'), \phi(t')) \Big) \right) \delta(\phi(t_i)) \mathcal{O}_{\mathrm{DS}}(\phi^*(t), \phi(t)) \tag{47}$$

In the limit $t_i \to -\infty$, where we have allowed the system to equilibrate by time t, we can discard the delta function enforcing the initial condition. Taking $t_f \to \infty$ as well yields

$$\boxed{\langle \mathcal{O}(t) \rangle = \int \mathcal{D}\phi \exp \left(- \int_{-\infty}^{\infty} dt' \Big(\phi^*(t')\partial_{t'}\phi(t') + H_{\mathrm{DS}}(\phi^*(t'), \phi(t')) \Big) \right) \mathcal{O}_{\mathrm{DS}}(\phi^*(t), \phi(t)).} \tag{48}$$

Without the Doi shift, we get an extra term $\phi(t_f)$ in the exponential corresponding to a final condition. However, in the limit $t_f \to \infty$, we can discard this term, and the non-Doi-shifted path integral takes the same form as Eq. (48) but without the Doi shift in the pseudo-Hamiltonian or observable.

Path integral for the extinction process

Let us return to the extinction process discussed earlier. To derive its pseudo-Hamiltonian, we start with its master equation (Eq. (7)), multiply both sides by $|n\rangle$, and sum over n, yielding

$$\partial_t |\psi(t)\rangle = \epsilon \left(\sum_{n=0}^{\infty} (n+1) P(n+1) |n\rangle - \sum_{n=0}^{\infty} n P(n) |n\rangle \right)$$

$$= \epsilon \sum_{n=0}^{\infty} P(n) \left(n |n-1\rangle - n |n\rangle \right) \tag{49}$$

$$= \epsilon \left(a - a^\dagger a \right) \sum_{n=0}^{\infty} P(n) |n\rangle . \tag{50}$$

We can read off the pseudo-Hamiltonian as

$$H(a^\dagger, a) = \epsilon (a^\dagger - 1) a. \tag{51}$$

As anticipated from our discussion of probability conservation, this pseudo-Hamiltonian satisfies $H(1, a) = 0$. Applying the Doi shift and switching to fields gives

$$H_{\mathrm{DS}}(\phi^*(t), \phi(t)) = \epsilon |\phi(t)|^2. \tag{52}$$

Finally, the corresponding path-integral action (see Eq. (48)) takes the form

$$S[\phi^*, \phi] = \int_{-\infty}^{\infty} dt \phi^*(t) (\partial_t + \epsilon) \phi(t). \tag{53}$$

Thus, particle extinction manifests as a mass term in the action. The propagator, which is the functional inverse of $K(t, t') = (\partial_t + \epsilon)\delta(t - t')$, is $\sim\Theta(t - t') e^{-\epsilon(t - t')}$.

We can extend this simple model by adding more complex terms to the original master equation, which introduces nonlinear interaction terms in the action. These can be handled using standard perturbation theory methods (e.g., Feynman diagrams). Alternatively, Buice and Cowan derive fluctuation effects within the path integral framework by deriving loop corrections to the effective action, given by the Legendre transformation of the log-path integral.